

BTZ Solutions on Codimension-2 Braneworlds

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Abstract. We consider five-dimensional gravity with a Gauss-Bonnet term in the bulk and an induced gravity term on a 2-brane of codimension-2. We show that this system admits BTZ-like black holes on the 2-brane which are extended into the bulk with regular horizons.

1. Introduction

Codimension-2 braneworlds, i.e, a brane embedded in a bulk with two extra dimensions, have recently stimulated a growing interest. The most attractive feature of these models is that the vacuum energy of the brane instead of curving the brane world-volume, merely induces a deficit angle in the bulk around the brane [1]. This property was used to solve the cosmological constant problem [2]. However, soon it was realized [3] that one can only find nonsingular solutions if the brane stress tensor is proportional to its induced metric. To obtain the Einstein equation on the brane one has to introduce a cut-off (brane thickness) [4], losing the predictability of the theory, or alternatively, one can add to the gravitational action a Gauss-Bonnet (GB) term [5] or a scalar curvature term (induced gravity) on the brane [6].

Time dependent cosmological solutions in this scenario are also a complex topic. In the thin brane limit, since the brane and bulk energy momentum tensors are related, we cannot get the standard cosmology on the brane [7]. Thus, one has to regularize the codimension-2 branes by introducing some thickness and then consider matter on them [8], arriving to a time-dependent bulk solution. Or alternatively, one can conceive a codimension-1 brane moving in the regularized static background [9]. However, the resulting cosmology is unrealistic having a negative Newton's constant.

Moreover, the issue of localization of a black hole on the brane and its extension to the bulk is not fully understood. In codimension-1 braneworlds, the most natural generalization of the four-dimensional Schwarzschild metric was to consider a black string infinite in the fifth dimension [10]. However, although the curvature scalars are everywhere finite, the Kretschmann scalar diverges at the AdS horizon at infinity, which turns the above solution into a physically unsuitable object. It has been argued that there exists a localized black cigar solution with a finite extension along the extra dimension due to a Gregory-Laflamme [11] type of instability near the AdS horizon, although no analytical solution has been found until the present time. Further attempts deal with the Einstein equations projected on the brane [12], which include an unknown bulk dependent term, the Weyl tensor projection. Due to this reason the system

is not closed, and some assumptions have to be made either in the form of the metric or in the Weyl term [13]. The stability and thermodynamics of these solutions were worked out in [14].

A lower dimensional version of a black hole living on a (2+1)-dimensional braneworld was considered in [15]. The authors based their analysis on the C-metric [16] modified by a cosmological constant term. After some coordinate transformations, the resulting geometry corresponds to a BTZ black hole [17] on the brane. This object can be extended as a BTZ string in a four-dimensional anti-de Sitter (AdS) bulk, which has been cut in order to avoid the conical singularity along the symmetry semi-axis. Their thermodynamical stability analysis showed that the black string remains a stable configuration when its transverse size is comparable to the four-dimensional AdS radius, being destabilized by the Gregory-Laflamme instability above that scale, breaking up to a BTZ black hole on a 2-brane.

In the codimension-2 scenario, a six-dimensional black hole on a 3-brane was proposed in [18]. This is a generalization of the 4D Aryal, Ford, Vilenkin [19] black hole pierced by a cosmic string adjusted to the codimension-2 branes with a conical structure in the bulk and deformations fitting the deficit angle. However, it is not clear how to realize these solutions in the thin 3-brane limit.

In this work we study black holes on an infinitely thin conical 2-brane and their extension into the five-dimensional bulk.

2. The Setup

We consider the following gravitational action in five dimensions with a Gauss-Bonnet (GB) term in the bulk and an induced three-dimensional curvature term on the brane

$$S_{\text{grav}} = \frac{M_5^3}{2} \left\{ \int d^5x \sqrt{-g^{(5)}} \left[R^{(5)} + \alpha \left(R^{(5)2} - 4R_{MN}^{(5)} R^{(5)MN} + R_{MNKL}^{(5)} R^{(5)MNKL} \right) \right] \right. \\ \left. + r_c^2 \int d^3x \sqrt{-g^{(3)}} R^{(3)} \right\} + \int d^5x \mathcal{L}_{\text{bulk}} + \int d^3x \mathcal{L}_{\text{brane}}, \quad (1)$$

where $\alpha (\geq 0)$ is the GB coupling constant, $r_c = M_3/M_5^3$ is the induced gravity “cross-over” scale, which marks the transition from 3D to 5D gravity, and M_5, M_3 are the five and three-dimensional Planck masses, respectively.

The above induced term has been written in the particular coordinate system in which the metric is

$$ds_5^2 = g_{\mu\nu}(x, \rho) dx^\mu dx^\nu + a^2(x, \rho) d\rho^2 + b^2(x, \rho) d\theta^2. \quad (2)$$

Here $g_{\mu\nu}(x, 0)$ is the brane metric, whereas x^μ denotes three dimensions, $\mu = 0, 1, 2$, and ρ, θ denote the radial and angular coordinates of the two extra dimensions (the ρ direction may or may not be compact, and the θ coordinate ranges from 0 to 2π). Capital M, N indices will take values in the five-dimensional space. Note that we have assumed that there exists an azimuthal symmetry in the system, so that both the induced three-dimensional metric and the functions a and b do not depend on θ .

The Einstein equations resulting from the variation of the action (1) are

$$G_M^{(5)N} + r_c^2 G_\mu^{(3)\nu} g_M^\mu g_\nu^N \frac{\delta(\rho)}{2\pi b} - \alpha H_M^N = \frac{1}{M_5^3} \left[T_M^{(B)N} + T_\mu^{(br)\nu} g_M^\mu g_\nu^N \frac{\delta(\rho)}{2\pi b} \right], \quad (3)$$

where

$$H_M^N = \left[\frac{1}{2} g_M^N (R^{(5)2} - 4R_{KL}^{(5)2} + R_{ABKL}^{(5)2}) - 2R^{(5)} R_M^{(5)N} \right. \\ \left. + 4R_{MP}^{(5)} R_{(5)}^{NP} + 4R_{KMP}^{(5)} R_{(5)}^{KPN} - 2R_{MKLP}^{(5)} R_{(5)}^{NKL P} \right]. \quad (4)$$

To obtain the braneworld equations we expand the metric around the brane as

$$b(x, \rho) = \beta(x)\rho + O(\rho^2) . \quad (5)$$

At the boundary of the internal two-dimensional space where the 2-brane is situated the function b behaves as $b'(x, 0) = \beta(x)$, where a prime denotes derivative with respect to ρ . We also demand that the space in the vicinity of the conical singularity is regular, which imposes the supplementary conditions, $\partial_\mu \beta = 0$ and $\partial_\rho g_{\mu\nu}(x, 0) = 0$ [5].

The extrinsic curvature in the particular gauge $g_{\rho\rho} = 1$ that we are considering is given by $K_{\mu\nu} = g'_{\mu\nu}$. The above decomposition will be helpful in the following for finding the induced dynamics on the brane. We will now use the fact that the second derivatives of the metric functions contain δ -function singularities at the position of the brane. The nature of the singularity then gives the following relations [5]

$$\frac{b''}{b} = -(1 - b') \frac{\delta(\rho)}{b} + \text{non-singular terms} , \quad (6)$$

$$\frac{K'_{\mu\nu}}{b} = K_{\mu\nu} \frac{\delta(\rho)}{b} + \text{non-singular terms} . \quad (7)$$

From the above singularity expressions and using the Gauss-Codazzi equations we can match the singular parts of the Einstein equations (3) and get the following “boundary” Einstein equations

$$G_{\mu\nu}^{(3)} = \frac{1}{M_{(5)}^3 (r_c^2 + 8\pi(1 - \beta)\alpha)} T_{\mu\nu}^{(br)} + \frac{2\pi(1 - \beta)}{r_c^2 + 8\pi(1 - \beta)\alpha} g_{\mu\nu} . \quad (8)$$

The effective three-dimensional Planck mass and cosmological constant are simply

$$M_{Pl}^2 = M_{(5)}^3 (r_c^2 + 8\pi(1 - \beta)\alpha) , \quad (9)$$

$$\Lambda_{(3)} = \lambda - 2\pi M_{(5)}^3 (1 - \beta) , \quad (10)$$

where λ is the brane tension. Note that the Planck mass can depend on the deficit angle. This is an effect of solely the bulk Gauss-Bonnet term.

As a result of the Gauss-Codazzi reduction, the above boundary Einstein equations will also contain terms proportional to the extrinsic curvature and additional terms coming from the bulk GB term. However, if we allow only regular conical singularities there is no contribution from these terms [5].

3. Black String Solutions

We assume that there is a localized (2+1) black hole on the brane described by the following metric

$$ds_3^2 = \left(-n(r)^2 dt^2 + n(r)^{-2} dr^2 + r^2 d\phi^2 \right) , \quad (11)$$

where $0 \leq r < \infty$ is the radial coordinate, and ϕ has the usual periodicity $(0, 2\pi)$. We will look for black string solutions of the Einstein equations (3) using the five-dimensional metric (2) in the form

$$ds_5^2 = f^2(\rho) \left(-n(r)^2 dt^2 + n(r)^{-2} dr^2 + r^2 d\phi^2 \right) + d\rho^2 + b^2(r, \rho) d\theta^2 , \quad (12)$$

where we have chosen $a(r, \rho) = 1$ without loss of generality.

The space outside the conical singularity is regular, therefore, we demand that the warp function $f(\rho)$ is also regular everywhere. We assume that there is only a cosmological constant Λ_5 in the bulk. Then, the bulk Einstein equations take the form

$$G_{MN}^{(5)} - \alpha H_{MN} = -\frac{\Lambda_5}{M_5^3} g_{MN} . \quad (13)$$

By combining the (rr) and $(\phi\phi)$ equations we get

$$\left(\dot{n}^2 + n\ddot{n} - \frac{n\dot{n}}{r}\right) \left(1 - 4\alpha \frac{b''}{b}\right) = 0, \quad (14)$$

while a combination of the $(\rho\rho)$ and $(\theta\theta)$ equations gives

$$\left(f'' - \frac{f'b'}{b}\right) \left[3 - 4\frac{\alpha}{f^2} \left(\dot{n}^2 + n\ddot{n} + 2\frac{n\dot{n}}{r} + 3f'^2\right)\right] = 0, \quad (15)$$

where a dot implies derivatives with respect to r . The solutions of the equations (14) and (15) are summarized in the following table [20].

$n(r)$	$f(\rho)$	$b(\rho)$	$-\Lambda_5$	Constraints
BTZ	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\forall b(\rho)$	$\frac{3}{4\alpha}$	$L_3^2 = 4\alpha$
BTZ	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$2\beta\sqrt{\alpha}\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{3}{4\alpha}$	-
BTZ	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$2\beta\sqrt{\alpha}\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{3}{4\alpha}$	$L_3^2 = 4\alpha$
BTZ	± 1	$\frac{1}{\gamma}\sinh(\gamma\rho)$	$\frac{3}{l^2}$	$\gamma = \sqrt{-\frac{2\Lambda_5}{3+4\alpha\Lambda_5}}$
$\forall n(r)$	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$2\beta\sqrt{\alpha}\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{3}{4\alpha}$	-
Corrected BTZ	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$2\beta\sqrt{\alpha}\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{3}{4\alpha}$	$L_3^2 = 4\alpha$
Corrected BTZ	± 1	$2\beta\sqrt{\alpha}\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{1}{4\alpha}$	$\Lambda_5 = -\frac{1}{4\alpha} = -\frac{3}{L_3^2}$

Table 1. BTZ String-Like Solutions in Five-Dimensional Braneworlds of Codimension-2

where L_3 is the length of three-dimensional AdS space. The BTZ solution is given by [17]

$$n^2(r) = -M + \frac{r^2}{L_3^2}. \quad (16)$$

This black hole for positive mass has a horizon at $r = L_3\sqrt{M}$, and the radius of curvature of the AdS_3 space $L_3 = (-\Lambda_3)^{-1/2}$ provides the length scale necessary to define this horizon. For the mass $-1 < M < 0$, which is dimensionless, the BTZ black hole has a naked conical singularity while for $M = -1$ the vacuum AdS_3 space is recovered.

Moreover, the “corrected” version of the BTZ black hole is found to be

$$n^2(r) = -M + \frac{r^2}{L_3^2} - \frac{\zeta}{r}. \quad (17)$$

This black hole corresponds to the BTZ black hole conformally coupled to a massless scalar field [21]. The scalar field does not introduce an independent conserved charge, for this reason it does not show up in the form of the metric. It merely modifies the energy-momentum tensor of the three-dimensional Einstein equations as we will see in the next section.

Although $n(r)$ remains undetermined in one of our solutions, it exists a connecting relation between this function and the matter on the brane $T_1^1 = T_2^2 = nn'/r + \Lambda_3$, $T_3^3 = n'^2 + nn'' + \Lambda_3$ from (8).

All these solutions extend the black hole on the brane into the bulk. In the BTZ case if we calculate the curvature invariants, we find no $r = 0$ curvature singularity for the BTZ string-like solution as expected. The warp function $f^2(\rho)$ gives the shape of the horizon of the string-like solution while the size of the horizon is defined by the scale $\sqrt{\alpha}$.

From our solutions we can notice that consistency of the five-dimensional bulk equations requires a fine-tuned relation between the GB coupling constant and the AdS_5 length (through the five-dimensional cosmological constant). The use of this fine-tuning gives to the non-singular horizon the shape of a throat up to the boundary of the AdS space.

4. Localization of the Black Hole on the Brane

In order to complete our solution with the introduction of the brane we must solve the corresponding junction conditions given by the “boundary” Einstein equations (8) using the induced metric on the brane given by (11). For the case when $n(r)$ corresponds to the BTZ black hole (16), and the brane cosmological constant is given by $\Lambda_{(3)} = -1/L_3^2$, we found that the energy-momentum tensor is null. Therefore, the BTZ black hole is localized on the brane in vacuum.

When $n(r)$ is of the form given in (17), we found the following traceless energy-momentum tensor

$$T_{\alpha}^{\beta} = \text{diag} \left(\frac{\zeta}{2r^3}, \frac{\zeta}{2r^3}, -\frac{\zeta}{r^3} \right), \quad (18)$$

which is conserved on the brane, $\nabla_{\beta} T_{\alpha}^{\beta} = 0$. This property has been studied in [22] for codimension-2 braneworlds in six dimensions; however, it seems to be a property independent of the bulk dimension for Einstein-Gauss-Bonnet codimension-2 models having an axially symmetric metric like (2) with $a^2(x, \rho) = 1$, and obeying junction conditions analogous to (8). If we consider the energy-momentum tensor in [21] necessary to sustain the solution (17), and we take the limit $r/L_3 \ll 1$, we get the unexpected result that it reduces to (18) which is necessary to localize the black hole on the conical 2-brane. A way to understand this result is that because in this limit r is very small, the black hole will be localized around the conical singularity and therefore, any matter will take a distributional form around this singularity. Note also that this solution is a result of the presence of the GB term in the bulk. If we switch off the GB coupling, then from relations (14) and (15) it can be seen that only the BTZ black hole is a solution.

5. Conclusions

We discussed black hole localization on an infinitely thin 2-brane of codimension-2 and its extension into a five-dimensional AdS bulk. To have a three-dimensional gravity on the brane we introduced a five-dimensional Gauss-Bonnet term in the bulk and an induced gravity term on the 2-brane. We showed that the (2+1) BTZ black hole and its short-distance-corrected extension can be localized on the 2-brane, while in the bulk these solutions describe BTZ strings. Consistency of the five-dimensional bulk equations requires a fine-tuned relation between the Gauss-Bonnet coupling constant and the length of the five-dimensional AdS space. The use of this fine-tuning gives to the non-singular horizon the shape of a throat up to the boundary of the AdS space.

We did not allow more severe singularities than conical. This assumption has fixed the deficit angle to a constant value.

This analysis has also been extended to a conical 3-brane embedded in a six-dimensional bulk as reported in [23]. This study also showed the relevance of the GB term as it dictates the kind of brane and bulk matter necessary to sustain a black hole on the brane.

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References

- [1] Chen J W, Luty M A and Ponton E, 2000, *JHEP* **0009**, 012.
- [2] Carroll S M, Guica M M, *Preprint* arXiv:hep-th/0302067; Aghababaie Y *et al.*, 2004, *Nucl. Phys. B* **680**, 389; Nilles H P, Papazoglou A, Tasinato G, 2004, *Nucl. Phys. B* **677**, 405; Kehagias A, 2004, *Phys. Lett. B* **600**, 133.
- [3] Cline J M, Descheneau J, Giovannini M, Vinet J, 2003, *JHEP* **0306**, 048.
- [4] Kanno S, Soda J, 2004, *JCAP* **0407**, 002. Vinet J, Cline J M, 2004, *Phys. Rev. D* **70**, 083514; Navarro I, Santiago J, 2005, *JHEP* **0502**, 007.
- [5] Bostock P, Gregory R, Navarro I and Santiago J, 2004, *Phys. Rev. Lett.* **92**, 221601.
- [6] Papantonopoulos E, Papazoglou A, 2005, *JCAP* **0507**, 004.
- [7] Kofinas G, 2006, *Phys. Lett. B* **633**, 141. Papantonopoulos E, Papazoglou A, 2005, *JHEP* **0509**, 012.
- [8] Carter B, Battye R A and Uzan J P, 2003, *Commun. Math. Phys.* **235**, 289; Kolanovic M, Porrati M and Rombouts J W, 2003, *Phys. Rev. D* **68**, 064018; Peloso M, Sorbo L and Tasinato G, 2006, *Phys. Rev. D* **73**, 104025; Papantonopoulos E, Papazoglou A and Zamarias V, 2007, *JHEP* **0703**, 002. Himmetoglu B and Peloso M, 2007, *Nucl. Phys. B* **773**, 84; Kobayashi T and Minamitsuji M, 2007, *Phys. Rev. D* **75**, 104013; Burgess C P, Hoover D and Tasinato G, 2007, *JHEP* **0709**, 124; Kobayashi T and Takamizu Y I, 2007, *Class. Quant. Grav.* **25**, 015007; Copeland E J and Seto O, 2007, *JHEP* **0708**, 001.
- [9] Papantonopoulos E, Papazoglou A and Zamarias V, 2008, *Nucl. Phys. B* **797**, 520-536; Minamitsuji M and Langlois D, 2007, *Phys. Rev. D* **76**, 084031.
- [10] Chamblin A, Hawking S W, Reall H S, 2000, *Phys. Rev. D* **61**, 065007.
- [11] Gregory R and Laflamme R, 1993, *Phys. Rev. Lett.* **70**, 2837.
- [12] Shiromizu T, Maeda K I and Sasaki M, 2000, *Phys. Rev. D* **62**, 024012.
- [13] Dadhich N, Maartens R, Papadopoulos P and Rezanian V, 2000, *Phys. Lett. B* **487**, 1; Bruni M, Germani C and Maartens R, 2001, *Phys. Rev. Lett.* **87**, 231302; Casadio R, Fabbri A and Mazzacurati L, 2002, *Phys. Rev. D* **65**, 084040; Kofinas G, Papantonopoulos E and Pappa I, 2002, *Phys. Rev. D* **66**, 104014; Kofinas G, Papantonopoulos E and Zamarias V, 2002, *Phys. Rev. D* **66**, 104028.
- [14] Abdalla E, Cuadros-Melgar B, Pavan A B and Molina C, 2006, *Nucl. Phys. B* **752**, 40.
- [15] Emparan R, Horowitz G T and Myers R C, 2000, *JHEP* **0001**, 007; Emparan R, Horowitz G T and Myers R C, 2000, *JHEP* **0001**, 021.
- [16] Kinnersley W and Walker M, 1970, *Phys. Rev. D* **2**, 1359.
- [17] Bañados M, Teitelboim C and Zanelli J, 1992, *Phys. Rev. Lett.* **69**, 1849.
- [18] Kaloper N and Kiley D, 2006, *JHEP* **0603**, 077. Kiley D, 2007, *Phys. Rev. D* **76**, 126002.
- [19] Aryal M, Ford L H and Vilenkin A, 1986, *Phys. Rev. D* **34**, 2263; Achúcarro A, Gregory R and Kuijken K, 1995, *Phys. Rev. D* **52**, 5729.
- [20] Cuadros-Melgar B, Papantonopoulos E, Tsoukalas M and Zamarias V, 2008, *Phys. Rev. Lett.* **100**, 221601.
- [21] Martinez C and Zanelli J, 1996, *Phys. Rev. D* **54**, 3830.
- [22] Kofinas G, 2005, *Class. Quant. Grav.* **22**, L47.
- [23] Cuadros-Melgar B, Papantonopoulos E, Tsoukalas M and Zamarias V, 2009, *Nucl. Phys. B* **810**, 246-265.